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ON V_0-STABILITY OF NUMERICAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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On V_0-stability of numerical methods for Volterra integral equations of the second kind

by

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ABSTRACT

Recently, WOLKENFELT [11] has shown that (ρ,σ)-reducible quadrature methods for Volterra integral equations of the second kind cannot be V_0-stable. This important class of discretization methods contains in particular the implicit Euler method and the trapezoidal method (which may be viewed as extended one-stage implicit Runge-Kutta methods of Pouzet type). The above negative result raises the question on the existence of V_0-stable methods for such equations. In the present note we derive a class of (extended) one-stage implicit Runge-Kutta methods of Bel'tyukov type which are V_0-stable, and we show that these methods are closely related to P-stable linear multistep methods for second-order initial-value problems.

KEY WORDS & PHRASES: *Volterra integral equations of the second kind, V_0-stability, Runge-Kutta methods of Pouzet and Bel'tyukov type, P-stability*

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1. INTRODUCTION

Consider the two-parameter family of Volterra integral equations of the second kind,

$$(1.1)\qquad y(t) = y_0 + \int_0^t \{\lambda+\mu(t-s)\}y(s)ds, \qquad t \geq 0,$$

with $\lambda < 0$, $\mu \leq 0$. Since (1.1) is equivalent to the second-order initial-value problem

$$(1.2)\qquad y'' = \lambda y' + \mu y, \quad y(0) = y_0, \quad y'(0) = \lambda y_0,$$

the exact solution of (1.1) may be written in the form

$$(1.3)\qquad y(t) = \frac{y_0 r_1}{r_1-r_2}\exp(r_1 t) + \frac{y_0 r_2}{r_2-r_1}\exp(r_2 t) \qquad (r_1 \neq r_2)$$

where $r_{1,2} := (\lambda \pm \sqrt{D(\lambda,\mu)})/2$, with

$$(1.4)\qquad D = D(\lambda,\mu) := \lambda^2 + 4\mu.$$

If $D < 0$ (with $\lambda < 0$, $\mu \leq 0$) then the exact solution is oscillatory (with period $4\pi/\sqrt{-D}$) and exponentially damped; for $\lambda = 0$, $\mu < 0$ we obtain $y(t) = y_0 \cos(t\sqrt{-\mu})$. On the other hand, if $D > 0$ ($\lambda < 0$, $\mu \leq 0$) then the coefficients in (1.3) have opposite signs and hence $y(t)$ has precisely one change of sign for some positive t and satisfies $y(t) \to 0$ as $t \to \infty$.

In the subsequent discussion we are particularly interested in the qualitative behavior of solutions $\{y_n\}$ ($h > 0$ fixed) of discretization methods for (1.1): while in the case $D > 0$ our interest lies with $\lim_{n\to\infty} y_n$ ($h > 0$ fixed), for $D < 0$ (damped periodic oscillations) we shall not only require that $\lim_{n\to\infty} y_n = 0$ ($h > 0$ fixed) but also that $\{y_n\}$ simulates the oscillatory part (for some period).

Ideally, a numerical method for (1.1) ought to preserve, for any given stepsize $h > 0$, the qualitative behavior of the exact solution (1.3) (except perhaps for an initial transition phase) described above; this means also that the method should be able to detect reasonably well whether the exact solution corresponds to $D > 0$ (no periodic oscillations; exponential decay for t sufficiently large), or to $D < 0$ (damped periodic oscillations).

The following concept was first introduced in connection with the stability analysis of numerical methods for Volterra integro-differential equations (compare [1], [8], [9]) and was adapted for Volterra integral equations (1.1) in [11], [10], [4]. Here, we make use of a slightly different terminology (compare the general remarks, regarding the above test equation (1.1) and the general concept of numerical stability for Volterra integral equations of the second kind one should eventually be aiming for, at the end of this section).

<u>DEFINITION</u>. A discretization method for Volterra integral equations of the second kind is said to be *V_0-stable* if it yields approximations $\{y_n\}$ for $\{y(t_n)\}$ ($t_n = nh$, $n = 1,2,\dots;h > 0$) which satisfy $y_n \to 0$ as $n \to \infty$ whenever it is applied, with fixed stepsize $h > 0$, to (1.1) with arbitrary $(\lambda,\mu) \in Q_{\lambda,\mu} := \{(\lambda,\mu): \lambda < 0, \mu \le 0\}$.

Recently, WOLKENFELT [11] has derived the surprising result that (ρ,σ)-reducible quadrature methods for Volterra integral equations of the second kind cannot be V_0-stable (compare this negative result with positive results on A_0-stability for Volterra integro-differential equations: [1], [8]!); this important class of discretization methods includes the implicit Euler method which for the general Volterra equation

$$(1.5)\qquad y(t) = g(t) + \int_0^t K(t,s,y(s))ds, \quad t \ge 0,$$

assumes the form

$$(1.6a)\qquad y_{n+1} = hK(t_{n+1},t_{n+1},y_{n+1}) + \tilde{F}_n(t_{n+1}) \quad (n \ge 1),$$

with

$$(1.6b)\qquad \tilde{F}_n(t) := h \sum_{j=1}^{n} K(t,t_j,y_j) + g(t), \quad t \ge t_n.$$

If we set

(1.7) $$Q(h\lambda,h^2\mu) := \{(h\lambda,h^2\mu): h\lambda < 0,\ h^2\mu \le 0\ (h > 0)\},$$

then the implicit Euler method applied to (1.1) yields approximations $\{y_n\}$ with $y_n \to 0$ as $n \to \infty$ ($h > 0$ fixed) provided that

$$(h\lambda,h^2\mu) \in Q(h\lambda,h^2\mu) \cap \{(h\lambda,h^2\mu): h^2\mu > 2h\lambda - 4\}.$$

This means that there exist values $(\lambda,\mu) \in Q_{\lambda,\mu}$ with $|\lambda|$ small and $\mu << 0$ for which $\{y_n\}$ does not remain bounded (compare [11], [10]; see also [4]).

Wolkenfelt's negative result (see also his conjecture for more general quadrature methods) raises the question of whether V_0-stable methods for (1.5) do exist at all and, if so, whether they include any of the "classical" (Runge-Kutta or linear multistep) methods. In the following section we present a class of one-stage implicit Runge-Kutta methods of Bel'tyukov type (see [2] for a description of Runge-Kutta methods for (1.5)) which are V_0-stable; it then turns out that this class is closely related to certain P-stable linear two-step methods for second-order initial-value problems (see [7]; also [3], [5] and [6]). In section 3 we discuss the qualitative behavior of $\{y_n\}$ for a number of one-step methods, and we make a conjecture on V_0-stability for Runge-Kutta methods of Pouzet type.

It should be emphasized that the analysis for (1.1) presented in this paper is only a preliminary step towards a comprehensive analysis of numerical stability for (linear) Volterra integral equations much more general than the special convolution equation (1.1): while (1.1) provides important new insight into the qualitative behavior of solutions of discretization methods which cannot be obtained from applying these methods to the "basic" test equation

$$y(t) = y_0 + \int_0^t \lambda\, y(s)ds \quad (\lambda < 0),$$

the eventual aim will be to produce an analysis of stability for *discrete* Volterra equations,

(1.8) $$y_n = g_n + \sum_{j=0}^{n} \kappa_{n,j} y_j \qquad (n \ge 0)$$

(where the discrete kernel $\{\kappa_{n,j}\}$ depends on the kernel of the given (linear) Volterra integral equation as well as on the discretization method); such an analysis will be closely modelled after the stability theory for Volterra integral equations of the second kind. (Compare also [9] for some preliminary results in this direction, especially for Volterra integro-differential equations).

However, at present the theory of discrete Volterra equations (1.8) is largely non-existent and will need substantial development before such a general stability analysis can be attempted.

2. A CLASS OF V_0-STABLE RUNGE-KUTTA-BEL'TYUKOV METHODS

Consider the following discretization method for the general Volterra equation (1.5)

$$y_{n+1} = hK(t_n+c_1h, t_n+d_1h, y_{n+1}) + \tilde{F}_n(t_n+d_1h) \qquad (n \geq 0), \tag{2.1}$$

where $c_1 \geq 1 \geq d_1$, and where $\tilde{F}_n(t)$ is given by (1.6b). While for $c_1 = d_1 = 1$ we have again the implicit Euler method (1.6) (which may be viewed as an extended one-stage implicit Runge-Kutta method of Pouzet type (see, for example, [2], for a discussion of various classes of Runge-Kutta methods)), the choice $c_1 > 1$ yields a Runge-Kutta method of Bel'tyukov type; for $d_1 = 1$ (the case we shall study in the following) the methods (1.6) and (2.1) possess the same discretization $\tilde{F}_n(t_{n+1})$ for the so-called *lag term* $\int_0^{t_n} K(t_{n+1}, s, y(s))ds$ in (1.5).

<u>THEOREM 1</u>. *The Runge-Kutta-Bel'tyukov method* (2.1) *(with* $d_1 = 1$*) is* V_0*-stable if, and only if,* $c_1 \geq 5/4$.

<u>PROOF</u>. If we apply (2.1) to (1.1) we find the recurrence relation

$$y_{n+1} = h\{\lambda+h\mu(c_1-1)\}y_{n+1} + h\sum_{j=1}^{n}\{\lambda+h\mu(n+1-j)\}y_j + y_0 \qquad (n \geq 0), \tag{2.2}$$

and this leads in a straightforward manner to

$$(2.3)\qquad y_{n+2} - 2y_{n+1} + y_n = h\lambda[y_{n+2}-y_{n+1}] + $$
$$+ h^2\mu[(c_1-1)y_{n+2} + (3-2c_1)y_{n+1} + (c_1-1)y_n] \quad (n \geq 0),$$

Define $\rho(z)$ and $\sigma(z)$ by

$$(2.4)\qquad \begin{aligned} \rho(z) &:= z^2 - 2z + 1 \quad (=(z-1)^2), \quad \text{and} \\ \sigma(z) &:= (c_1-1)(z^2 + \frac{3-2c_1}{c_1-1} z + 1) \quad (c_1 > 1). \end{aligned}$$

We note that if $\lambda = 0$ then the choice $c_1 = 1$ in (2.3) yields the (explicit) two-step method of Störmer for (1.2). If $c_1 > 1$ (and $\lambda = 0$) we are faced with an *implicit* two-step method for (1.2) whose characteristic polynomials (2.4) are both symmetric (and, for $c_1 = 5/4$, are given by

$$\rho(z) = (z-1)^2 \quad \text{and} \quad \sigma(z) = (\tfrac{1}{2}(z+1))^2;$$

in this context we also refer to [3]).

The characteristic equation associated with (2.3) is

$$(2.5)\qquad \begin{aligned} \tau(z;h\lambda,h^2\mu) &:= \rho(z) - h\lambda(z^2-z) - h^2\mu\sigma(z) \\ &:= a_2z^2 + a_1z + a_0 = 0 \end{aligned}$$

(with $a_i = a_i(h\lambda,h^2\mu)$). It follows from the Routh-Hurwitz criterion that the roots of (2.5) lie inside the unit circle if and only if

(i) $a_0 + a_1 + a_2 > 0 \quad (\Rightarrow -h^2\mu < 0)$,
(ii) $2(a_2 - a_0) > 0 \quad (\Rightarrow -2h\lambda > 0)$, and
(iii) $a_0 + a_2 - a_1 > 0$,

which yields the condition

$$(2.6)\qquad h^2\mu(5-4c_1) > 2h\lambda - 4 \quad (h\lambda < 0,\ h^2\mu \leq 0).$$

Hence we deduce from (2.6) that the method (2.1) is V_0-stable if, and only if, $5 - 4c_1 \leq 0$. □

For the limiting case $\lambda = 0$ ($\mu < 0$) in (2.3) we have a linear two-step method for the initial-value problem $y'' = \mu y$, $y(0) = y_0$, $y'(0) = 0$. We thus have the following

<u>COROLLARY</u>. *The linear two-step method for* $y'' = f(t,y)$, $y(0) = y_0$, $y'(0) = z_0$,

$$y_{n+2} - 2y_{n+1} + y_n = h^2\{(c_1-1)f_{n+2}+(3-2c_1)f_{n+1}+(c_1-1)f_n\} \quad (n \geq 0) \tag{2.7}$$

is P-*stable (in the sense of* [7]) *if, and only if,* $c_1 \geq 5/4$.

The proof is obvious and therefore omitted.

We note, however, that the above one-parameter family (2.7) ($c_1 \geq 5/4$) of P-stable methods has been given by LAMBERT and WATSON ([7], Method II on p.198); in their notation it reads

$$y_{n+2} - 2y_{n+1} + y_n = h^2\{\frac{1}{2-2\cos\phi} f_{n+2} - \frac{\cos\phi}{1-\cos\phi} f_{n+1} + \frac{1}{2-2\cos\phi} f_n\} \quad (n \geq 0), \tag{2.8}$$

where $0 < \phi < 2\pi$. The methods (2.8) and (2.7) are equivalent since

$$F(\phi) := (3-2\cos\phi)/(2-2\cos\phi)$$

maps $(0,2\pi)$ onto $[5/4,\infty)$, with $F(\pi) = 5/4$.

As has already been pointed out before, the characteristic polynomials ρ and σ of (2.7) are both perfect squares (of first-degree polynomials) if, and only if, $c_1 = 5/4$; thus for this limiting value of c_1 (with regard to V_0-stability), (2.7) is equivalent to a linear one-step method (namely, the trapezoidal method) for the first-order system corresponding to (1.2) ($\lambda=0$). (Compare also [3], [6] for an analysis of this special linear multistep method for $y'' = \mu y$, $\mu < 0$.)

3. THE QUALITATIVE BEHAVIOR OF $\{y_n\}$ FOR $\lambda < 0$, $\mu < 0$

Even though the property of V_0-stability of a discretization method for (1.5) guarantees that, for $h > 0$ fixed, $y_n \to 0$ as $n \to \infty$ whenever the method is applied to (1.1) with $(\lambda,\mu) \in Q_{\lambda,\mu}$, one might ask whether that method yields approximations $\{y_n\}$ which truly simulate the qualitative behavior of the exact solution (indicated in Fig. 1) already for (sufficiently large) finite values of n.

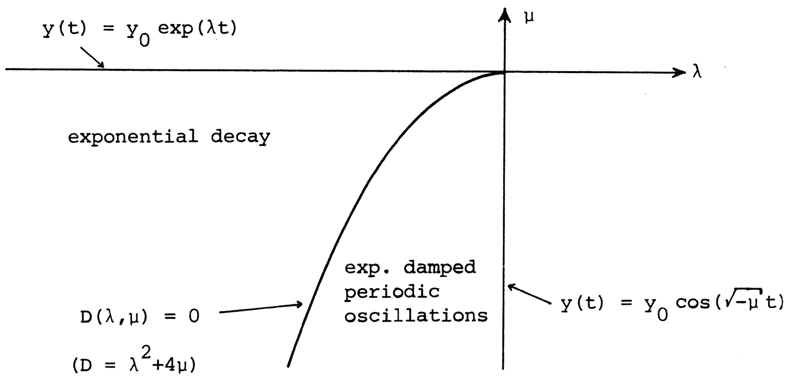


Fig. 1

In other words: given $\lambda < 0$ and $\mu < 0$ for which we have either $D(\lambda,\mu) > 0$, or $D(\lambda,\mu) < 0$ (i.e. the exact solution is an exponentially damped *periodic* oscillation), how well is a V_0-stable discretization method capable of detecting the corresponding qualitative nature of the exact solution? Since the qualitative behavior of the numerical solution $\{y_n\}$ depends on the form of the roots of a certain characteristic equation associated with the method, we illustrate the above question by first listing these characteristic equations for four simple but important discretization methods for (1.5), including the V_0-stable Runge-Kutta-Bel'tyukov method (2.1) ($c_1 \geq 5/4$).

THEOREM 2. *The recurrence relations resulting from applying the explicit Euler method, the implicit Euler method, the trapezoidal method, and the Runge-Kutta-Bel'tyukov method* (2.1) ($d_1 = 1$) *(with all methods used in extended form) to the Volterra equation* (1.1) *reduce, respectively, to the* *following second-order difference equations:*

(3.1) $$y_{n+2} - 2y_{n+1} + y_n = h\lambda(y_{n+1}-y_n) + h^2\mu\, y_{n+1};$$

(3.2) $$y_{n+2} - 2y_{n+1} + y_n = h\lambda(y_{n+2}-y_{n+1}) + h^2\mu\, y_{n+1};$$

(3.3) $$y_{n+2} - 2y_{n+1} + y_n = \frac{h\lambda}{2}(y_{n+2}-y_n) + h^2\mu\, y_{n+1}; \quad \text{and}$$

(3.4) $$y_{n+2} - 2y_{n+1} + y_n = h\lambda(y_{n+2}-y_{n+1}) + h^2\mu[(c_1-1)y_{n+2}+(3-2c_1)y_{n+1}+(c_1-1)y_n] \quad (c_1 \geq 1).$$

Note that for $\lambda = 0$ the difference equations for the two methods of Euler and the trapezoidal method are all identical to the (explicit) two-step Störmer method.

Since the characteristic equations associated with (3.1)-(3.4) have the form

(3.5) $$a_2(h\lambda,h^2\mu)\, z^2 + a_1(h\lambda,h^2\mu)\, z + a_0(h\lambda,h^2\mu) = 0,$$

the solutions of the difference equations (and hence of the given discretization methods) may be written as

(3.6) $$y_n = \sum_{j=1}^{2} \gamma_j\, (z_j(h\lambda,h^2\mu))^n \quad (n \geq 0),$$

where z_1, z_2 denote the roots (assumed to be distinct for ease of exposition) of (3.5).

Let

(3.7) $$\Delta = \Delta(h\lambda,h^2\mu) := a_1^2 - 4a_0a_2;$$

since $a_i = a_i(h\lambda,h^2\mu) \in \mathbb{R}$ $(i = 0,1,2)$, the character of the approximations (3.6) will depend on the sign of Δ, in analogy to the qualitative behavior of $y(t_n)$ (see (1.3), (1.1)).

In the following we set $u := h\lambda$, $v := h^2\mu$ (with $u < 0$, $v \leq 0$).

<u>THEOREM 3</u>. *The discriminants* $\Delta = \Delta_p(h\lambda, h^2\mu)$ *in* (3.7) *for the four discretization methods mentioned in Theorem* 2 *are given by*

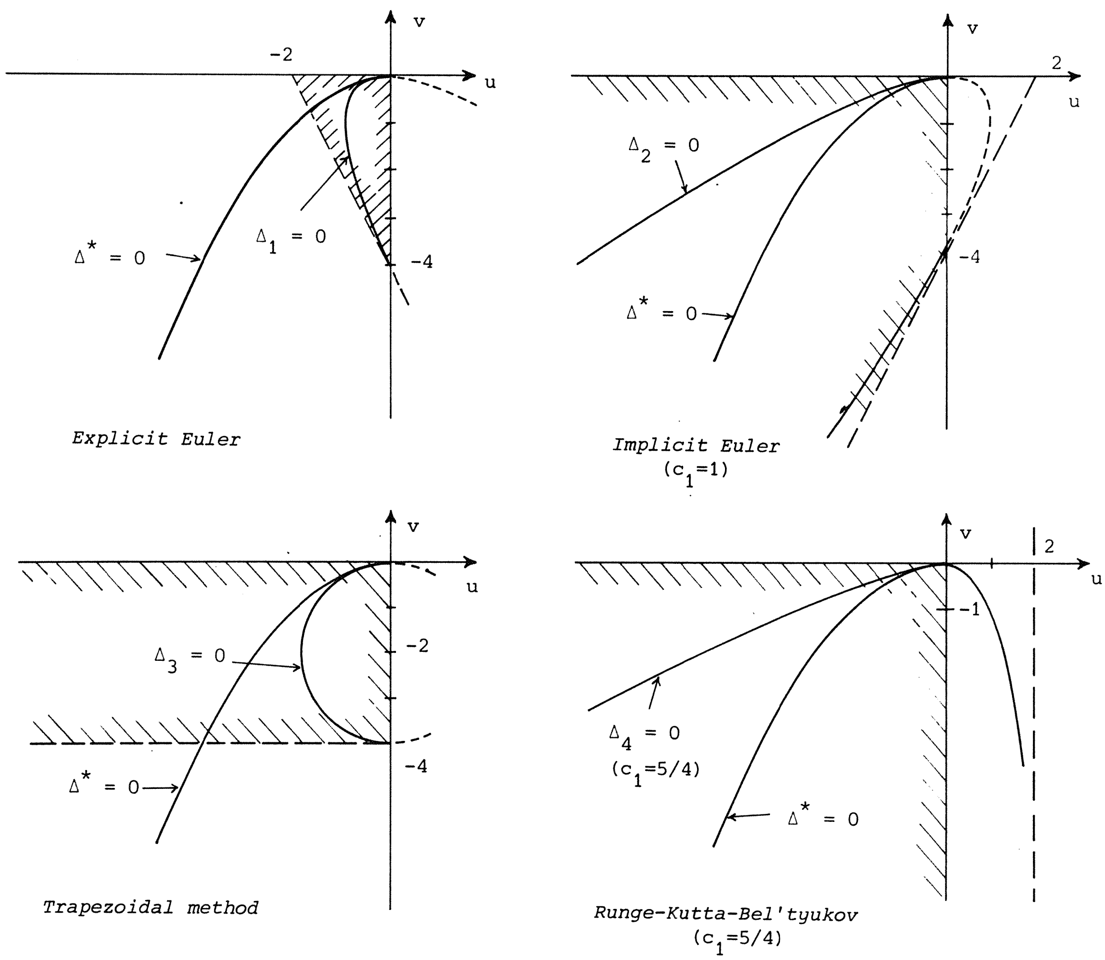
(3.8) $\Delta_1 = (u+v)^2 + 4v$ (parabola: axis ∥ $v = -u$);

(3.9) $\Delta_2 = (u-v)^2 + 4v$ (parabola: axis ∥ $v = u$);

(3.10) $\Delta_3 = u^2 + v^2 + 4v$ (circle); and

(3.11) $\Delta_4 = (u-v)^2 + 4(1-c_1)v^2 + 4v$ (hyperbola, if $c_1 > 1$).

The graphs of these conics are given in Fig. 2 below, together with that of $\Delta^* := u^2 + 4v$ $(=h^2 D(\lambda,\mu))$. In addition, we indicate the corresponding regions of absolute stability for these methods.



<u>Fig. 2</u>

We observe that, for $(u,v) = (h\lambda, h^2\mu)$ in the respective region of absolute stability,

(i) both the explicit Euler method and the trapezoidal method yield still damped *non-periodic* approximations $\{y_n\}$ while the exact solution is already characterized by $D(\lambda,\mu) < 0$; and

(ii) the implicit Euler method and its Bel'tyukov type modification (here shown for $c_1 = 5/4$) produce damped *periodic* oscillations before the exact solution reaches this qualitative stage.

4. REMARKS AND CONJECTURES

We conclude this note wil some additional remarks and conjectures. First, it is clear that even though a given method (like the Runge-Kutta-Bel'tyukov methods of Theorem 1) may be V_0-stable it will be of interest to construct V_0-stable discretization methods for which the corresponding difference equation has solutions which better simulate the exact solution (i.e. the curve $\Delta = 0$ should be close to the curve $\Delta^* = 0$). Second, it was shown in [11] that both the implicit Euler method and the trapezoidal method (used in extended form) are not V_0-stable; both methods may be interpreted as Runge-Kutta-Pouzet methods. In view of this result and the special structure of these Runge-Kutta methods (note that, if applied to (1.1), the diagonal terms in the Runge-Kutta equations, i.e. $ha_{ii}K(t_n+c_ih, t_n+c_ih, Y_i^{(n)})$ $(i = 1,\ldots,m)$, all vanish for $\lambda = 0$), we conjecture that *implicit Runge-Kutta methods of (extended) Pouzet type cannot be V_0-stable.*

Finally, we state a second conjecture, namely: if an *extended* Runge-Kutta method (of Pouzet or of Bel'tyukov type) is not V_0-stable, then any corresponding *mixed* Runge-Kutta method (i.e. a Runge-Kutta method in which the approximation to the exact lag term of (1.5),

$$F_n(t) := \int_0^{t_n} K(t,s,y(s))ds + g(t) \qquad (t \geq t_n),$$

is based on quadrature processes which are *not* based on the given Runge-Kutta parameters), then again the method cannot be V_0-stable. In other words, the forward step (i.e. the discretization of $\int_{t_n}^{t} K(t,s,y(s))ds$, $t \in (t_n, t_{n+1}]$)

in an extended or mixed Runge-Kutta method, is the most crucial one for V_0-stability.

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REFERENCES

[1] BRUNNER, H. & J.D. LAMBERT, Stability of numerical methods for Volterra integro-differential equations, *Computing*, 12 (1974), 75-89.

[2] BRUNNER, H. & S.P. NØRSETT, Runge-Kutta theory for Volterra integral equations of the second kind, Mathematics and Computation No. 1/80, Dept. of Mathematics, University of Trondheim, 1980.

[3] DAHLQUIST, G., On accuracy and unconditional stability of linear multistep methods for second order differential equations, *BIT*, 18 (1978), 133-136.

[4] HOUWEN, P.J. van der, Convergence and stability analysis of Runge-Kutta type methods for Volterra integral equations of the second kind, Report NW 83/80, Mathematisch Centrum, Amsterdam.

[5] JAIN, M.K., R.K. JAIN & U.A. KRISHNAIAH, P-stable methods for periodic initial value problems of second order differential equations, *BIT*, 19 (1979), 347-355.

[6] JELTSCH, R., Stability on the imaginary axis and A-stability of linear multistep methods, *BIT*, 18 (1978), 170-174.

[7] LAMBERT, J.D. & I.A. WATSON, Symmetric multistep methods for periodic initial value problems, *J. Inst. Math. Appl.*, 18 (1976), 189-202.

[8] MATTHYS, J., A-stable linear multistep methods for Volterra integro-differential equations, *Numer. Math.*, 27 (1976), 85-94.

[9] NEVANLINNA, O., Positive quadratures for Volterra equations, *Computing*, 16 (1976), 349-357.

[10] WILLIAMS, H., S. MCKEE & H. BRUNNER, The numerical stability and relative merits of multistep methods for convolution Volterra integral equations, to appear.

[11] WOLKENFELT, P.H.M., Stability analysis of reducible quadrature methods for Volterra integral equations of the second kind, Report NW 79/80, Mathematisch Centrum, Amsterdam, 1980.